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On two different bi-Hamiltonian structures for the Toda lattice

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Abstract
We discuss two different incompatible Poisson pencils for the Toda lattice by using known variables of separation proposed by Moser and by Sklyanin.

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1. Introduction

A bi-Hamiltonian manifold M is a smooth manifold endowed with two compatible bi-vectors P_0, P_1 such that

$$[P_0, P_0] = [P_0, P_1] = [P_1, P_1] = 0,$$

where $[.,.]$ is the Schouten bracket. Such a condition assures that the linear combination $P_0 - \lambda P_1$ is a Poisson pencil, i.e. it is a Poisson bi-vector for each $\lambda \in \mathbb{C}$, and therefore the corresponding bracket $\{.,.\}_0 + \lambda\{.,.\}_1$ is a pencil of Poisson brackets [5].

Dynamical systems on M having enough functionally independent integrals of motion $H_1, \ldots, H_n$ in the involution with respect to the both Poisson brackets

$$\{H_i, H_j\}_0 = \{H_i, H_j\}_1 = 0 \tag{1.1}$$

will be called the bi-integrable systems.

The main aim of this paper is to prove that the Toda lattice is a bi-integrable system with respect to two essentially different Poisson pencils $P_0 + \lambda P_1$ and $P_0 + \lambda P_1^\star$, which are related with two known families of the separated variables [6, 7].

The first Poisson pencil $P_0 + \lambda P_1$ related to the action–angle or the Moser variables was found by Das, Okubo and Fernandes [1, 4]. For the periodic Toda lattice similar Poisson tensor in physical variables was found in [9]. Construction of the second Poisson pencil $P_0 + \lambda P_1^\star$ related to the Sklyanin variables is a new result.

2. The separation of variables method

Let us briefly recall some necessary facts about the separation of variables method [2, 3, 8, 9].

A complete integral $S(q,t)$ of the Hamilton–Jacobi equation

$$\frac{\partial S(q,t)}{\partial t} + H\left(q, \frac{\partial S(q,t)}{\partial q}, t\right) = 0, \tag{2.1}$$

where $q = (q_1, \ldots, q_n)$, is a solution $S(q, t, \alpha_1, \ldots, \alpha_n)$ depending on n parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that

$$\det \left\| \frac{\partial^2 S(q,t,\alpha)}{\partial q_i \partial \alpha_j} \right\| \neq 0. \tag{2.2}$$

For any complete integral of (2.1) solutions $q_i = q_i(t, \alpha, \beta)$ and $p_i = p_i(t, \alpha, \beta)$ of the Hamilton equations of motion may be found from the Jacobi equations

$$\beta_i = -\frac{\partial S(q,t,\alpha)}{\partial \alpha_i}, \qquad p_i = \frac{\partial S(q,t,\alpha)}{\partial q_i}, \qquad i = 1, \ldots, n. \tag{2.3}$$

According to (2.2) if we resolve a second part of the Jacobi equations with respect to parameters $\alpha_1, \ldots, \alpha_n$ one gets n independent integrals of motion

$$\alpha_m = H_m(p, q, t), \qquad m = 1, \ldots, n, \tag{2.4}$$

as functions on the phase space M with coordinates p, q.

Definition 1. *A dynamical system is a separable system if the corresponding complete integral $S(q,t,\alpha)$ has an additive form*

$$S(q, t, \alpha_1, \ldots, \alpha_n) = -Ht + \sum_{i=1}^{n} S_i(q_i, \alpha_1, \ldots, \alpha_n). \tag{2.5}$$

Here the ith component S_i depends only on the ith coordinate q_i and α.

In such a case the Hamiltonian H is said to be separable and coordinates q are said to be separated coordinates for H, in order to stress that the possibility of finding an additive complete integral of (2.5) depends on the choice of the coordinates.

For the separable dynamical system we have

$$\frac{\partial^2}{\partial q_k \partial q_j} S = \frac{\partial}{\partial q_k}\left(\frac{\partial}{\partial q_j} S_j\right) = 0, \qquad \text{for all} \qquad j \neq k, \tag{2.6}$$

such that the second Jacobi equations (2.3) are the separated equations

$$p_j = \frac{\partial}{\partial q_j} S_j(q_j, \alpha_1, \ldots, \alpha_n) \qquad \text{or} \qquad \phi_j(p_j, q_j, \alpha) = p_j - \frac{\partial}{\partial q_j} S_j(q_j, \alpha) = 0. \tag{2.7}$$

Here the jth equation contains a pair of canonical variables p_j and q_j only.

Proposition 1. *For any separable dynamical system integrals of motion $H_m(p,q,t)$ (2.4) are in the involution*

$$\{H_k, H_m\}_f = 0, \qquad k, m = 1, \ldots, n,$$

with respect to the following brackets $\{.,.\}_f$ on M:

$$\{p_i, q_j\}_f = \delta_{ij} f_j(p,q), \qquad \{p_i, p_j\}_f = \{q_i, q_j\}_f = 0, \tag{2.8}$$

which depend on n arbitrary functions $f_1(p,q), \ldots, f_n(p,q)$.

Proof. In fact, we have to repeat the proof of the Jacobi theorem given by Liouville. Namely, differentiate relations (2.4) by q_j and then substitute momenta p_k from separated equations (2.7) to obtain

$$\frac{\partial H_m}{\partial q_j} + \sum_{i=1}^{n} \frac{\partial H_m}{\partial p_i}\frac{\partial p_i}{\partial q_j} = \left(\frac{\partial H_m}{\partial q_j} + \frac{\partial H_m}{\partial p_j}\frac{\partial^2 S_j}{\partial q_j^2}\right) = 0.$$

□

It follows that for any H_k and H_m

$$\sum_{j=1}^{n} f_j \frac{\partial H_k}{\partial p_j}\left(\frac{\partial H_m}{\partial q_j} + \frac{\partial H_m}{\partial p_j}\frac{\partial^2 S_j}{\partial q_j^2}\right) = \sum_{j=1}^{n} f_j \frac{\partial H_k}{\partial p_j}\frac{\partial H_m}{\partial q_j} + \sum_{j=1}^{n} f_j \frac{\partial H_k}{\partial p_j}\frac{\partial H_m}{\partial p_j}\frac{\partial^2 S_j}{\partial q_j^2} = 0.$$

Permuting indices k and m and subtracting the resulting equation from the previous one we get

$$\sum_{j=1}^{n} f_j \left(\frac{\partial H_k}{\partial p_j}\frac{\partial H_m}{\partial q_j} - \frac{\partial H_m}{\partial p_j}\frac{\partial H_k}{\partial q_j}\right) = 0.$$

The final assertion easily follows.

Brackets $\{.,.\}_f$ (2.8) are the Poisson brackets if and only if

$$[P_f, P_f] = 0, \tag{2.9}$$

where

$$P_f = \begin{pmatrix} 0 & \mathrm{diag}(f_1,\ldots,f_n) \\ -\mathrm{diag}(f_1,\ldots,f_n) & 0 \end{pmatrix}. \tag{2.10}$$

Proposition 2. *If the jth function*

$$f_j(p,q) = f_j(p_j,q_j), \qquad j = 1,\ldots,n \tag{2.11}$$

depends only on the jth pair of coordinates p_j, q_j then brackets $\{.,.\}_f$ (2.8) are the Poisson brackets, which are compatible with canonical ones.

Proof. Substituting the tensor P_f (2.10) into the equations:

$$[P_0, P_f] = [P_f, P_f] = 0, \qquad \text{where} \qquad P_0 = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix},$$

one gets the following system of partial differential equations:

$$\frac{\partial f_j}{\partial q_k} = \frac{\partial f_j}{\partial p_k} = 0, \qquad f_i\frac{\partial f_j}{\partial q_k} = f_i\frac{\partial f_j}{\partial p_k} = 0,$$

for all $j \neq k \neq i$. The separable functions $f_j(p_j,q_j)$ (2.11) satisfy this system of equations. So, the corresponding tensor P_f (2.10) is the Poisson tensor, which is compatible with the canonical tensor P_0. □

According to the following proposition, the separation of variables method is closely related with the bi-Hamiltonian geometry [2, 3].

Proposition 3. *Any separable dynamical system is bi-integrable system.*

Proof. In order to get a pair of compatible Poisson brackets on the phase space M it is enough to postulate brackets (2.8)

$$\{p_i, q_j\}_f = \delta_{ij} f_j(p_j,q_j), \qquad \{p_i, p_j\}_f = \{q_i, q_j\}_f = 0 \tag{2.12}$$

between the known separated variables q_j and p_j for a given dynamical system. According to proposition 1, the corresponding integrals of motion $H_m(p,q,t)$ (2.4) are in the bi-involution with respect to brackets $\{.,.\}_0$ and $\{.,.\}_f$. □

For the dynamical systems separable in the so-called Darboux–Nijenhuis coordinates proposition 3 has been proved in [2, 3] using the recursion operator $N = P_f P_0^{-1}$ and its geometric properties.

The separated variables (p,q) are defined up to canonical transformations

$$p_j \to \tilde{p}_j = X_j(p_j,q_j), \qquad q_j \to \tilde{q}_j = Y_j(p_j,q_j), \tag{2.13}$$

which have to preserve the canonical tensor P_0 and would change the form of the separated equations $\phi_j(p_j,q_j,\alpha) = 0$ (2.7) and the second tensor P_f. It is clear that freedom in the choice of the functions $f_j(p_j,q_j)$ is related with this freedom in the definition of the separated variables.

Let us consider the dynamical system simultaneously separable in the (p,q) and $(\tilde{p},\tilde{q})$ variables related by the generic canonical transformation

$$\begin{aligned} p_j &\to \tilde{p}_j = X_j(p_1,\ldots,p_n,q_1,\ldots,q_n),\\ q_j &\to \tilde{q}_j = Y_j(p_1,\ldots,p_n,q_1,\ldots,q_n), \end{aligned} \tag{2.14}$$

which is distinguished from the particular transformation (2.13). For such systems we can suppose the following:

Proposition 4. *Dynamical systems separable in the (p,q) and $(\tilde{p},\tilde{q})$ variables related by (2.14) are bi-integrable systems with respect to two different Poisson pencils.*

It is clear that there are a lot of systems separable in different separated variables canonically related to each other, because a family of transformations (2.14) include standard transformations from the separated variables (p,q) to the action–angle variables $(\tilde{p},\tilde{q})$. Another example is the so-called superintegrable systems.

In the next sections, we demonstrate how these propositions work for the open and periodic Toda lattices.

Remark 1. According [2, 3] for the stationary systems differentiating the separated equations (2.7)

$$\left(\frac{\partial\phi_j}{\partial q_j}\,\mathrm{d}q_j + \frac{\partial\phi_j}{\partial p_j}\,\mathrm{d}p_j\right) + \sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}\,\mathrm{d}H_i = 0, \tag{2.15}$$

then applying $N^* = P_0^{-1}P_f$ and substituting (2.15) into the resulting equation one gets

$$f_j\left(\frac{\partial\phi_j}{\partial q_j}\,\mathrm{d}q_j + \frac{\partial\phi_j}{\partial p_j}\,\mathrm{d}p_j\right) + \sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}N^*\mathrm{d}H_i = -f_j\sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}\,\mathrm{d}H_i + \sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}N^*\mathrm{d}H_i = 0.$$

It follows that

$$\sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}N^*\mathrm{d}H_i = f_j\sum_{i=1}^{n}\frac{\partial\phi_j}{\partial H_i}\,\mathrm{d}H_i, \qquad j = 1,\ldots,n,$$

that is, in matrix form

$$N^*\mathrm{d}H = \mathrm{d}H\,F. \tag{2.16}$$

Here, an $n\times n$ control matrix F with eigenvalues $f_1,\ldots,f_n$ is defined by

$$F = J^{-1}\,\mathrm{diag}(f_1,\ldots,f_n)J, \qquad J_{ji} = \frac{\partial\phi_j}{\partial H_i}$$

and $\mathrm{d}H$ is a $2n\times n$ matrix with entries $\mathrm{d}H_{ij} = \partial H_j/\partial z_i$, where $z = (q,p)$.

Equation (2.16) means that the subspace spanned by covectors $\mathrm{d}H_1, \ldots, \mathrm{d}H_n$ is invariant with respect to N^* [3].

Example 1. For further use we introduce the special control matrix F, which is the Frobenius matrix

$$F_f = \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \cdots & 0 & 1 \\ c_n & 0 & \cdots & & 0 \end{pmatrix}. \tag{2.17}$$

Here, c_k are the coefficients of the characteristic polynomial $\Delta_N(\lambda)$ of the recursion operator $N = P_f P_0^{-1}$:

$$\Delta_N(\lambda) = (\det(N - \lambda \mathrm{I}))^{1/2} = \lambda^n - (c_1\lambda^{n-1} + \cdots + c_n) = \prod_{j=1}^{n}(\lambda - \lambda_j). \tag{2.18}$$

3. Open Toda lattice

We start this section listing some well-known facts about the open Toda lattice associated with the root system of $\mathscr{A}_n$ type.

The Hamilton function is equal to

$$H = \frac{1}{2}\sum_{i=1}^{n} {p_i}^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}},$$

where q_i denotes the position of the ith particle and p_i is its momenta such that

$$\{q_i, p_j\}_0 = \delta_{ij}, \qquad \{p_i, p_j\}_0 = \{q_i, q_j\}_0 = 0. \tag{3.1}$$

Consequently, the equations of motion read

$$\dot{q}_i = p_i, \qquad \dot{p}_1 = -\mathrm{e}^{q_1 - q_2}, \qquad \dot{p}_n = \mathrm{e}^{q_{n-1} - q_n}$$

and

$$\dot{p}_i = \mathrm{e}^{q_{i-1} - q_i} - \mathrm{e}^{q_i - q_{i+1}}, \qquad i = 2, \ldots, n-1.$$

The exact solution is due to the existence of a Lax matrix. Consider the L-operator

$$L_i = \begin{pmatrix} \lambda - p_i & -\mathrm{e}^{q_i} \\ \mathrm{e}^{-q_i} & 0 \end{pmatrix} \tag{3.2}$$

and the monodromy matrix

$$T(\lambda) = L_1(\lambda)\cdots L_{n-1}(\lambda)L_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \qquad \det T(\lambda) = 1, \tag{3.3}$$

which depends polynomially on the parameter λ:

$$T(\lambda) = \begin{pmatrix} \lambda^n + A_1\lambda^{n-1} + \cdots + A_n & B_1\lambda^{n-1} + \cdots + B_n \\ C_1\lambda^{n-1} + \cdots + C_n & D_2\lambda^{n-2} + \cdots + D_n \end{pmatrix}.$$

The monodromy matrix satisfies Sklyanin's Poisson brackets:

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\}_0 = [r(\lambda - \mu), T(\lambda) \otimes T(\mu)], \tag{3.4}$$

where $r(\lambda-\mu)$ is the standard 4×4 rational r-matrix:

$$r(\lambda-\mu)=\frac{-1}{\lambda-\mu}\Pi,\qquad \Pi=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{3.5}$$

The Monodromy matrix $T(\lambda)$ is the Lax matrix for the periodic Toda lattice, whereas the Lax matrix for the open Toda lattice is equal to

$$T_o(\lambda)=KT(\lambda)=\begin{pmatrix}A&B\\0&0\end{pmatrix}(\lambda),\qquad K=\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

The trace of the Lax matrix

$$\text{‘tr}\, T_o(\lambda)=A(\lambda)=\lambda^n+H_1\lambda^{n-1}+\cdots+H_n,\qquad \{H_i,H_j\}=0 \tag{3.6}$$

generates n independent integrals of motion H_i in the involution providing complete integrability of the system [7].

3.1. The Moser variables

In this section, we briefly discuss the relation between the Moser variables [6] and the known bi-Hamiltonian structure for the open Toda lattice given by Das and Okubo [1]. The more detailed discussion may be found in [9].

According to [6], we introduce the n pairs of the separated variables λ_i,μ_i, $i=1,\dots,n$, having the standard Poisson brackets,

$$\{\lambda_i,\lambda_j\}_0=\{\mu_i,\mu_j\}_0=0,\qquad \{\lambda_i,\mu_j\}_0=\delta_{ij}, \tag{3.7}$$

with the λ variables being n zeros of the polynomial $A(\lambda)$ and the μ variables being values of the polynomial $B(\lambda)$ at those zeros,

$$A(\lambda_i)=0,\qquad \mu_i=\ln B(\lambda_i),\qquad i=1,\dots,n. \tag{3.8}$$

The corresponding separated equations

$$A(\lambda_i)=(\lambda^n+H_1\lambda^{n-1}+\cdots+H_n)_{\lambda=\lambda_i}=0$$

depend on the coordinates λ_i only.

The interpolation data (3.8) and n identities

$$B(\lambda_i)C(\lambda_i)=\det T(\lambda)=1$$

allow us to construct the separation representation for the whole monodromy matrix $T(\lambda)$:

$$\begin{aligned}
A(\lambda)&=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n),\\
B(\lambda)&=A(\lambda)\sum_{i=1}^{n}\frac{\mathrm{e}^{\mu_i}}{(\lambda-\lambda_i)A'(\lambda_i)},\\
C(\lambda)&=-A(\lambda)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\mu_i}}{(\lambda-\lambda_i)A'(\lambda_i)},\\
D(\lambda)&=\frac{1+B(\lambda)C(\lambda)}{A(\lambda)}.
\end{aligned} \tag{3.9}$$

If we postulate the following second Poisson brackets (2.12)

$$\{\lambda_i,\mu_j\}_1=\lambda_i\delta_{ij},\qquad \{\lambda_i,\lambda_j\}_1=\{\mu_i,\mu_j\}_1=0,$$

one gets [9]

$$\begin{aligned} &\{A(\lambda), A(\mu)\}_1 = \{B(\lambda), B(\mu)\}_1 = 0, \\ &\{A(\lambda), B(\mu)\}_1 = \frac{1}{\lambda - \mu}(\mu A(\lambda)B(\mu) - \lambda A(\mu)B(\lambda)). \end{aligned} \tag{3.10}$$

The first bracket in (3.10) guarantees that integrals of motion H_i (3.6) from $A(\lambda)$ are in the bi-involution.

Substitute polynomials $A(\lambda)$ and $B(\lambda)$ in initial (p, q) variables into the brackets (3.10) and solve the resulting equations to obtain the known second Poisson tensor [1, 4]

$$P_1^{\text{open}} = \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i<j}^{n} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}. \tag{3.11}$$

The minimal characteristic polynomial of the corresponding recursion operator $N_M = P_1^{\text{open}} P_0^{-1}$ is equal to

$$\Delta_{N_M}(\lambda) = A(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j) = \lambda^n + \sum_{j=1}^{n} H_j \lambda^{n-j}.$$

So, the (λ, μ) coordinates are variables of the separation of the action–angle type [3], i.e. the corresponding dynamical equations are trivial:

$$\{H_i, \lambda_j\} = 0, \qquad i, j = 1, \ldots, n.$$

The Hamiltonians H_i (3.6) from $A(\lambda)$ satisfy the Frobenius recursion relations

$$N_M^* \, \mathrm{d}H_i = \mathrm{d}H_{i+1} - H_i \, \mathrm{d}H_1, \tag{3.12}$$

where $N_M^* = P_0^{-1} P_1^{\text{open}}$ and $H_{n+1} = 0$, i.e. The Hamiltonians H_i satisfy equation (2.16) with the Frobenius matrix F_f (2.17).

Example 2. For the three-particle open Toda lattice Hamiltonians H_i (3.6) are

$$\begin{aligned} &H_1 = -(p_1 + p_2 + p_3), \\ &H_2 = p_1 p_2 + p_1 p_3 + p_2 p_3 - \mathrm{e}^{q_1 - q_2} - \mathrm{e}^{q_2 - q_3}, \\ &H_3 = -p_1 p_2 p_3 + p_1 \mathrm{e}^{q_2 - q_3} + p_3 \mathrm{e}^{q_1 - q_2}. \end{aligned}$$

It is obvious that $H = H_1^2/2 - H_2$. The second Poisson tensor P_1 (3.11) in the matrix form reads

$$P_1^{\text{open}} = \begin{pmatrix} 0 & -1 & -1 & p_1 & 0 & 0 \\ 1 & 0 & -1 & 0 & p_2 & 0 \\ 1 & 1 & 0 & 0 & 0 & p_3 \\ -p_1 & 0 & 0 & 0 & -\mathrm{e}^{q_1 - q_2} & 0 \\ 0 & -p_2 & 0 & \mathrm{e}^{q_1 - q_2} & 0 & -\mathrm{e}^{q_2 - q_3} \\ 0 & 0 & -p_3 & 0 & \mathrm{e}^{q_2 - q_3} & 0 \end{pmatrix}. \tag{3.13}$$

The control matrix F in (2.16) is the Frobenius matrix

$$F_M^{\text{open}} = \begin{pmatrix} -H_1 & 1 & 0 \\ -H_2 & 0 & 1 \\ -H_3 & 0 & 0 \end{pmatrix}, \tag{3.14}$$

where the coefficients $c_i = -H_i$ coincide with integrals of motion.

3.2. The Sklyanin variables

In this section, we consider another known family of the separated variables, which give rise to the new bi-Hamiltonian structure for the open Toda lattice.

According to [7, 8] we can consider another set of the separated coordinates, which are poles of the Baker–Akhiezer function $\vec{\Psi}$ associated with the Lax matrix $T(\lambda)$ (3.3)

$$T(\lambda)\vec{\Psi} = u(\lambda)\vec{\Psi}, \qquad (\vec{\Psi}, \vec{\alpha}) = 1,$$

having the standard normalization $\vec{\alpha} = (1, 0)$.

In this case, the first half of variables is coming from $(n-1)$ finite roots and logarithm of leading coefficient of the non-diagonal entry of the monodromy matrix

$$B(\lambda) = -\mathrm{e}^{u_n} \prod_{j=1}^{n-1} (\lambda - v_j). \tag{3.15}$$

Another half is given by

$$u_j = -\ln A(v_j), \quad j = 1, \dots, n-1, \qquad \text{and} \qquad v_n = \sum_{i=1}^{n} p_i. \tag{3.16}$$

These equations (3.16) are the separated equations.

In these separated variables other entries of $T(\lambda)$ read

$$A(\lambda) = \left(\lambda + \sum_{j=1}^{n} v_j\right) \prod_{j=1}^{n-1} (\lambda - v_j) + \sum_{j=1}^{n-1} \mathrm{e}^{-u_j} \prod_{i \neq j}^{n-1} \frac{\lambda - v_i}{v_j - v_i} \tag{3.17}$$

and

$$D(\lambda) = -\sum_{j=1}^{n-1} \mathrm{e}^{u_j} \prod_{i \neq j}^{n-1} \frac{\lambda - v_i}{v_j - v_i}, \qquad C(\lambda) = \frac{A(\lambda)D(\lambda) - 1}{B(\lambda)}.$$

If we postulate the second Poisson brackets (2.12)

$$\{v_i, u_j\}_1^\star = v_i \delta_{ij}, \qquad \{v_i, v_j\}_1^\star = \{u_i, u_j\}_1^\star = 0$$

we gets

$$\{A(\lambda), A(\mu)\}_1^\star = \{B(\lambda), B(\mu)\}_1^\star = 0 \tag{3.18}$$

and

$$\{A(\lambda), B(\mu)\}_1^\star = \frac{1}{\lambda - \mu}(\lambda A(\lambda)B(\mu) - \mu A(\mu)B(\lambda)) + \mathrm{e}^{-u_n} \left(\lambda + \mu + \sum_{i=1}^{n-1} v_i\right) B(\lambda)B(\mu). \tag{3.19}$$

In initial (p, q) variables the last bracket looks like

$$\{A(\lambda), B(\mu)\}_1^\star = \frac{1}{\lambda - \mu}(\lambda A(\lambda)B(\mu) - \mu A(\mu)B(\lambda)) + \mathrm{e}^{-q_n} \left(\lambda + \mu + \sum_{i=1}^{n-1} p_i\right) B(\lambda)B(\mu). \tag{3.20}$$

The first bracket in (3.18) guarantees that integrals of motion H_i (3.6) from $A(\lambda)$ are in the bi-involution.

Substitute into the brackets (3.18)–(3.20) polynomials $A(\lambda)$ and $B(\lambda)$ in initial (p, q) variables and solve the resulting equations to obtain the following Poisson tensor:

$$P_1^\star = \sum_{i=1}^{n-2} \mathrm{e}^{q_i - q_{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n-1} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i<j}^{n-1} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}$$
$$+ \sum_{i=1}^{n-1} (p_i + p) \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_i} + \sum_{i=2}^{n-1} (\mathrm{e}^{q_i - q_{i+1}} - \mathrm{e}^{q_{i-1} - q_i}) \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial p_i}$$
$$+ p \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n} + \mathrm{e}^{q_1 - q_2} \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_n} + \mathrm{e}^{q_{n-2} - q_{n-1}} \frac{\partial}{\partial p_{n-1}} \wedge \frac{\partial}{\partial p_n}, \tag{3.21}$$

where $p = \sum_{i=1}^n p_i$ is a total momentum.

The tensor $P_1^\star$ is independent of q_n and the minimal characteristic polynomial of the corresponding recursion operator $N_S = P_1^\star P_0^{-1}$ is equal to

$$\Delta_{N_S} = -\mathrm{e}^{q_n} (\lambda + p) B(\lambda).$$

The normalized traces of the powers of N_S are integrals of motion for $(n-1)$-particle open Toda lattice. As a consequence, the Hamiltonians H_i (3.6) from $A(\lambda)$ satisfy equation (2.16) with the following control matrix:

$$F_S^{\text{open}} = \begin{pmatrix} -p & 0 \\ 0 & F_M^{\text{open}} \end{pmatrix}, \tag{3.22}$$

where F_M^{open} is the Frobenius matrix (2.17) associated with the recursion operator N_M for $(n-1)$-particle open Toda lattice.

The corresponding separated equations follow directly from the definitions of (u, v) variables (3.16):

$$\mathrm{e}^{-u_j} - A(v_j) = 0, \quad j = 1, \ldots, n-1 \qquad \text{and} \qquad v_n - \alpha_1 = 0, \tag{3.23}$$

where $A(\lambda) = \lambda^n + \sum_{i=1}^{n-1} \alpha_i \lambda^{n-i}$ and $\alpha_i = H_i$ are the values of integrals of motion. The first $(n-1)$ separated equations give rise to the equations of motion

$$\{A(\lambda), v_j\} = A(v_j) \prod_{i \neq j}^{n-1} \frac{\lambda - v_i}{v_j - v_i}, \qquad j = 1, \ldots, n-1, \tag{3.24}$$

which are linearized by the Abel transformation [7]

$$\left\{ A(\lambda), \sum_{k=1}^{n-1} \int^{v_k} \sigma_j \right\} = -\lambda^{j-1}, \qquad \sigma_j = \frac{\lambda^{j-1}\, \mathrm{d}\lambda}{A(\lambda)}, \qquad j = 1, \ldots, n-1,$$

where $\{\sigma_j\}$ is a basis of Abelian differentials of first order on an algebraic curve $z = A(\lambda)$ corresponding to the separated equations (3.23).

Remark 2. From the factorization of the monodromy matrix $T(\lambda)$ (3.3) one gets

$$T_n(\lambda) = T_{n-1}(\lambda) L_n(\lambda), \qquad \Rightarrow \qquad B_n(\lambda) = -\mathrm{e}^{q_n} A_{n-1}(\lambda),$$

where $B_n(\lambda)$ is the entry of the monodromy matrix $T_n(\lambda)$ of n-particle Toda lattice and $A_{n-1}(\lambda)$ is the entry of the monodromy matrix $T_{n-1}(\lambda)$ of $(n-1)$-particle Toda lattice.

This implies that for the $(n-1)$-particle chain the Moser variables λ_j coincide with the $n-1$ Sklyanin variables u_j, $i = 1, \ldots, n-1$, for the n-particle chain.

Example 3. At $n = 3$, the Poisson tensor $P_1^\star$ in the matrix form reads

$$P_1^\star = \begin{pmatrix} 0 & -1 & 0 & p_1 & 0 & -p_1 - p \\ 1 & 0 & 0 & 0 & p_2 & -p_2 - p \\ 0 & 0 & 0 & 0 & 0 & p \\ -p_1 & 0 & 0 & 0 & -e^{q_1-q_2} & e^{q_1-q_2} \\ 0 & -p_2 & 0 & e^{q_1-q_2} & 0 & -e^{q_1-q_2} \\ p_1 + p & p_2 + p & p & -e^{q_1-q_2} & e^{q_1-q_2} & 0 \end{pmatrix}$$

and at $n = 4$ it looks like

$$P_1^\star = \begin{pmatrix} 0 & -1 & -1 & 0 & p_1 & 0 & 0 & -p_1 - p \\ 1 & 0 & -1 & 0 & 0 & p_2 & 0 & -p_2 - p \\ 1 & 1 & 0 & 0 & 0 & 0 & p_3 & -p_3 - p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p \\ -p_1 & 0 & 0 & 0 & 0 & -e^{q_1-q_2} & 0 & e^{q_1-q_2} \\ 0 & -p_2 & 0 & 0 & e^{q_1-q_2} & 0 & -e^{q_2-q_3} & e^{q_2-q_3} - e^{q_1-q_2} \\ 0 & 0 & -p_3 & 0 & 0 & e^{q_2-q_3} & 0 & e^{q_2-q_3} \\ p_1 + p & p_2 + p & p_3 + p & p & -e^{q_1-q_2} & -e^{q_2-q_3} + e^{q_1-q_2} & -e^{q_2-q_3} & 0 \end{pmatrix}.$$

The corresponding control matrices F in (2.16) are given by

$$F_S^{\text{open}} = \begin{pmatrix} -p_1 - p_2 - p_3 & 0 & 0 \\ 0 & p_1 + p_2 & 1 \\ 0 & -p_1 p_2 + e^{q_1-q_2} & 0 \end{pmatrix} \tag{3.25}$$

and

$$F_S^{\text{open}} = \begin{pmatrix} -p_1 - p_2 - p_3 - p_4 & 0 & 0 & 0 \\ 0 & p_1 + p_2 + p_3 & 1 & 0 \\ 0 & -p_1p_2 - p_1p_3 - p_2p_3 + e^{q_1-q_2} - e^{q_2-q_3} & 0 & 1 \\ 0 & p_1p_2p_3 - p_1e^{q_2-q_3} - p_3e^{q_1-q_2} & 0 & 0 \end{pmatrix}. \tag{3.26}$$

3.3. Periodic Toda lattice

In this section, we discuss two different bi-Hamiltonian structures for the periodic Toda lattice.

The trace of the Lax matrix $T(\lambda)$ (3.3) for the periodic Toda lattice

$$\operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda) = \lambda^n + H_1\lambda^{n-1} + \cdots + H_n, \qquad \{H_i, H_j\} = 0 \tag{3.27}$$

generates n independent integrals of motion H_i in the involution providing complete integrability of the periodic Toda lattice [7]. For instance, the Hamilton function is equal to

$$H = \frac{1}{2}\sum_{i=1}^{n} {p_i}^2 + \sum_{i=1}^{n-1} e^{q_i-q_{i+1}} + e^{q_n-q_1}.$$

The Moser (λ, μ) variables (3.8) are the separated variables for the open Toda lattice only. Nevertheless, the bi-Hamiltonian structure for the periodic Toda lattice may be constructed by using generic properties of the Sklyanin bracket (3.4) [10]. According to [10], in order to get the second Poisson tensor P_1^{per} for the periodic Toda lattice we have to apply canonical transformation

$$p_1 \to p_1 + e^{-q_1}, \qquad p_n \to p_n + e^{q_n}. \tag{3.28}$$

to the tensor P_1^{open} (3.11) for the open Toda lattice . In this case, the Hamiltonians (3.27) satisfy equation (2.16) with the standard Frobenius matrix F_f (2.17) associated with the recursion operator $N_M^{\text{per}} = P_1^{\text{per}} P_0^{-1}$ [10].

The Sklyanin (u, v) variables (3.15), (3.16) are the separated variables for the open and periodic Toda lattices simultaneously. Therefore, the one Poisson tensor $P_1^{\star}$ determines the bi-Hamiltonian structure for both the Toda lattices, but the corresponding Hamiltonians satisfy equation (2.16) with the slightly different control matrices F_S^{open} and F_S^{per}.

Namely, for the open Toda lattice the control matrix F_S^{open} is given by (3.22). For the periodic Toda lattice we have to change the first column of this control matrix F_S^{open} (3.22) by the rule

$$F_{1,1}^{\text{per}} = -p, \qquad F_{i,1}^{\text{per}} = \left(p + \sum_{j=1}^{n-1} p_j\right) F_{i,2}^{\text{open}} + F_{i+1,2}^{\text{open}}, \qquad i = 2, \dots, n, \tag{3.29}$$

where $F_{n+1,2}^{\text{open}} = 0$.

Remark 3. For the open and periodic Toda lattices we have two matrix equations for the same integrals of motion:

$$N_M^* \mathrm{d}H = \mathrm{d}H\, F_M, \qquad N_S^* \mathrm{d}H = \mathrm{d}H\, F_S.$$

So, we have various complimentary relations

$$N_M^* N_S^*\, \mathrm{d}H = \mathrm{d}H\, F_M F_S \qquad \text{or} \qquad N_S^* (N_M^*)^{-1}\, \mathrm{d}H\, F_M = \mathrm{d}H\, F_S$$

between integrals of motion H_i, recursion operators N_S, N_M and the corresponding control matrices F_M, F_S.

Example 4. For the three-particle periodic Toda lattice Hamiltonians H_i (3.27) are

$$\begin{aligned} H_1 &= -(p_1 + p_2 + p_3), \\ H_2 &= p_1 p_2 + p_1 p_3 + p_2 p_3 - \mathrm{e}^{q_1 - q_2} - \mathrm{e}^{q_2 - q_3} - \mathrm{e}^{q_3 - q_1}, \\ H_3 &= -p_1 p_2 p_3 + p_1\, \mathrm{e}^{q_2 - q_3} + p_2\, \mathrm{e}^{q_3 - q_1} + p_3\, \mathrm{e}^{q_1 - q_2}. \end{aligned}$$

The second Poisson tensor (3.13) after canonical transformation (3.28) reads

$$P_1^{\text{per}} = \begin{pmatrix} 0 & -1 & -1 & \mathrm{e}^{-q_1} + p_1 & 0 & \mathrm{e}^{q_3} \\ 1 & 0 & -1 & \mathrm{e}^{-q_1} & p_2 & \mathrm{e}^{q_3} \\ 1 & 1 & 0 & \mathrm{e}^{-q_1} & 0 & p_3 + \mathrm{e}^{q_3} \\ -\mathrm{e}^{-q_1} - p_1 & -\mathrm{e}^{-q_1} & -\mathrm{e}^{-q_1} & 0 & -\mathrm{e}^{q_1 - q_2} & \mathrm{e}^{q_3 - q_1} \\ 0 & -p_2 & 0 & \mathrm{e}^{q_1 - q_2} & 0 & -\mathrm{e}^{q_2 - q_3} \\ -\mathrm{e}^{q_3} & -\mathrm{e}^{q_3} & -p_3 - \mathrm{e}^{q_3} & -\mathrm{e}^{q_3 - q_1} & \mathrm{e}^{q_2 - q_3} & 0 \end{pmatrix}.$$

The corresponding control matrix F in (2.16) is the Frobenius matrix

$$F_M^{\text{per}} = \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3 & 0 & 0 \end{pmatrix}, \tag{3.30}$$

where

$$\begin{aligned} c_1 &= p_1 + \mathrm{e}^{-q_1} + p_2 + p_3 + \mathrm{e}^{q_3}, \\ c_2 &= -p_1 p_2 - p_1 p_3 - p_2 p_3 - (p_2 + p_3)\, \mathrm{e}^{-q_1} - (p_1 + p_2)\, \mathrm{e}^{q_3} + \mathrm{e}^{q_1 - q_2} + \mathrm{e}^{q_2 - q_3} - \mathrm{e}^{q_1 - q_3}, \\ c_3 &= ((p_3 + \mathrm{e}^{q_3}) p_2 - \mathrm{e}^{q_2 - q_3}) p_1 + (p_3 + \mathrm{e}^{q_3}) p_2\, \mathrm{e}^{-q_1} - p_3\, \mathrm{e}^{q_1 - q_2} - \mathrm{e}^{-q_1 + q_2 - q_3} - \mathrm{e}^{q_1 - q_2 + q_3} \end{aligned}$$

are the coefficients of the minimal characteristic polynomial of the corresponding recursion operator $N_M^{\rm per} = P_1^{\rm per} P_0^{-1}$.

The same integrals of motion H_i for the periodic Toda lattice satisfy equation (2.16) with another recursion operator $N_S^* = P_0^{-1} P_1^\star$ and with the following control matrix (3.22)–(3.29):

$$F_S^{\rm per} = \begin{pmatrix} -p_1 - p_2 - p_3 & 0 & 0 \\ {\rm e}^{q_1-q_2} - p_1p_2 + (p_1+p_2)(2p_1+2p_2+p_3) & p_1+p_2 & 1 \\ ({\rm e}^{q_1-q_2} - p_1p_2)(2p_1+2p_2+p_3) & {\rm e}^{q_1-q_2} - p_1p_2 & 0 \end{pmatrix}. \qquad (3.31)$$

4. Conclusion

For the Toda lattice associated with the root system of $\mathscr{A}_n$ type we present two different bi-Hamiltonian structures on $M \simeq \mathbb{R}^{2n}$. The introduced Poisson tensors P_1 (3.11) and $P_1^\star$ (3.21) are incompatible

$$[P_1, P_1^\star] \neq 0,$$

while the corresponding recursion operators $N_M = P_1 P_0^{-1}$ and $N_S = P_1^\star P_0^{-1}$ take the diagonal form in the Moser and the Sklyanin separated variables, respectively.

Some relations between two different recursion operators N_M and N_S are briefly discussed in remark 3. It will be interesting to clarify these relations with the geometric point of view. For instance one would be interested to know what is the meaning of the (non-Poisson) tensor $P_1 - P_1^\star$, which appears by transformation of the separated variables to the action–angle variables.

Associated with the tensor P_1 brackets (3.10) were rewritten in the r-matrix form in [10]. It will be interesting to get the similar r-matrix formulation for the brackets (3.18)–(3.20) associated with the tensor $P_1^\star$.

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